Grade 7/8 Math Circles

February 9, 2022

Ancient Mathematics

Roman Numerals

You might not know it, but the digits 0, 1, 2, 3, ..., 9 that we use in our daily lives are referred to as **Hindu-Arabic numerals**. However they are only one way that we can represent quantities, to count items, and perform arithmetic. What are some other ways that we enumerate? Tally marks? Binary?

The title of this section alludes to another numeral system: **Roman numerals**. Although they might have served many more uses in Ancient Rome, these symbols are nowhere near as familiar as our Hindu-Arabic system. You may still encounter Roman numerals when reading books or plays, studying historical figures, or telling time on an analog clock.

The symbols used in this ancient numerical system are given below.

1	**5**	**10**	**50**	**100**	**500**	**1000**
I	V	X	L	C	D	M

Stop and Think

Why might the values of 5, 10, and their multiples have been chosen as the symbols? Think about how we teach young children how to count to 5 or 10.

Now let's see how these symbols can express familiar numbers.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
I	II	III	IV	V	VI	VII	VIII	IX

10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**
X	XX	XXX	XL	L	LX	LXX	LXXX	XC

100	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**
C	CC	CCC	CD	D	DC	DCC	DCCC	CM

There are a couple things that we can observe.

- There is no symbol for **0** (zero) in this numeral system.
- A symbol is never written more than three consecutive times.

Reading and Writing Roman Numerals

The two rules for reading Roman Numerals are:

1. When a symbol appears **after** a **larger** (or equal) symbol, it is **added**.
2. When a symbol appears **before** a **larger** symbol, it is **subtracted**.

Example 1

Determine the values of the Roman Numerals VII, XXVIII, IV, and CIX using the given rules.

Solution

VII and XXVIII both follow Rule 1 since the smaller symbols all come after the larger ones.

$$\text{VII} = \text{V} + \text{I} + \text{I} = 5 + 1 + 1 = 7$$

$$\text{XXVIII} = \text{X} + \text{X} + \text{V} + \text{I} + \text{I} + \text{I} = 10 + 10 + 5 + 1 + 1 + 1 = 28$$

IV has a smaller symbol before a larger one. As such, we use Rule 2's subtraction.

$$\text{IV} = \text{V} - \text{I} = 5 - 1 = 4$$

CIX has a smaller symbol I both after a larger C *and* before a larger X. In instances like these, Rule 2 takes precedence, meaning we must **prioritize subtracting** I from X and then add their difference to C.

$$\text{CIX} = \text{C} + (\text{X} - 1) = 100 + (10 - 1) = 100 + 9 = 109$$

Exercise 1

Determine the value each Roman Numeral represents.

(a) DC

(b) CLXVII

(c) XCIV

To convert a number into Roman Numerals:

1. Break the number into its thousands, hundreds, tens, and ones values.
2. Convert each value to its combination of Roman symbols.
3. Combine the symbols in order of decreasing value.

Example 2

Express 2143 using Roman Numerals.

Solution

1. $2143 = 2000 + 100 + 40 + 3$
2. 2000 is MM, 100 is C, 40 is XL, and 3 is III
3. Therefore, 2143 can be written as MMCXLIII

Exercise 2

Express each using Roman Numerals.

(a) 75

(b) 2022

(c) Your age

The Ancient Romans were hardly the first people to come up with the concept of numbers. The Hindu-Arabic Numerals are technically more ancient, given that both sets of symbols experienced many changes.

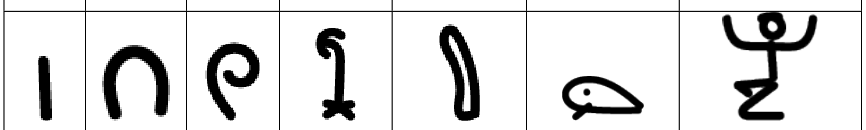
Ancient Egyptian Mathematics

The Ancient Egyptians from nearly 5000 years ago were some of the first mathematicians. While they didn't use the same formulas that we do, they had still developed a rather sophisticated number system and mathematical techniques to solve practical problems.

The problems from Ancient Egypt were certainly less complex than the ones we study in modern mathematics. But given that they had very little to work with, we admire their problem solving ability when it came to arithmetic and geometry. How lucky are we that the works of the Egyptians were preserved on delicate Papyrus, allowing us presently to study their craft.

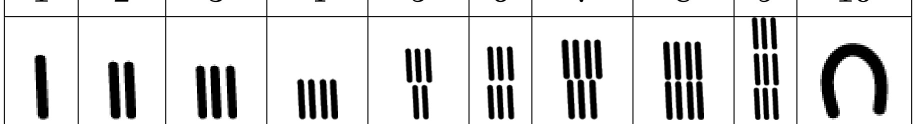
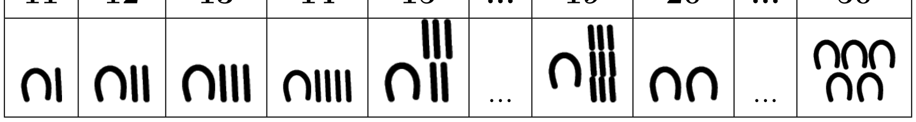
We will learn how they performed arithmetic and interpreted fractions, starting of course with how they expressed numerical values.

Hieroglyphic Numbers

1	10	100	1000	10 000	100 000	1 000 000
						
line	loop	rope	flower	finger	tadpole	god

It might take a bit of time to get used to hieroglyphs (e.g. one might not think of a flower to look like *that*, a god may look different for everyone, etc.). If you're having trouble with copying the symbols provided, draw your own, ensuring that it still matches its general description.

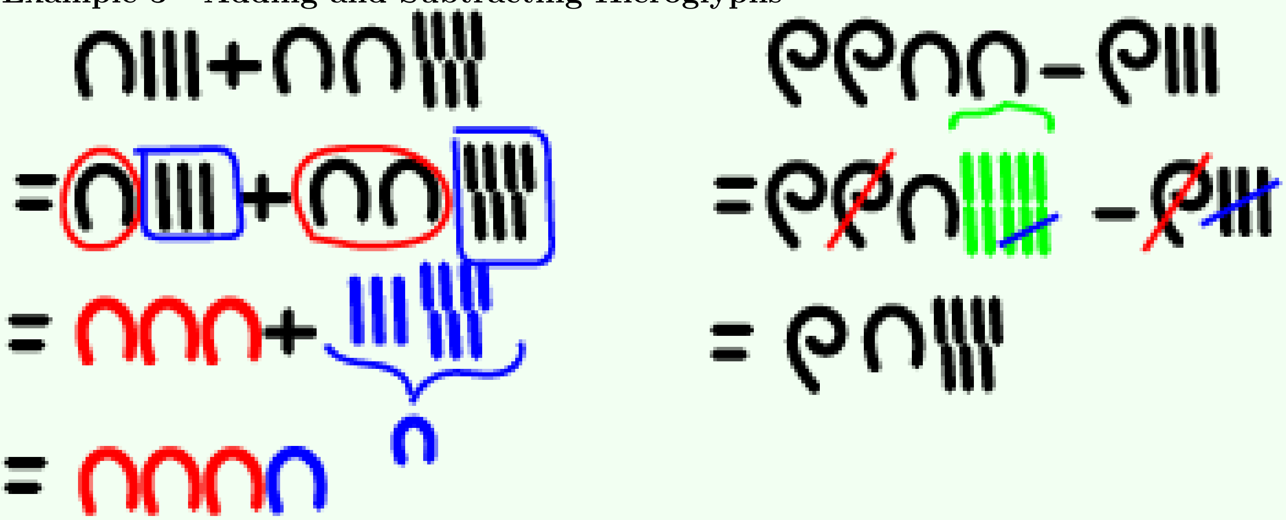
Similar to what we saw with Roman Numerals, we can combine these symbols to express more precise values. Some examples are shown below.

1	2	3	4	5	6	7	8	9	10
									
11	**12**	**13**	**14**	**15**	**...**	**19**	**20**	**...**	**50**
									

Translating back and forth from Hindu-Arabic to Ancient Egyptian is simpler. For reading hieroglyphs, add up all of the symbolic values. For drawing them, break up the digits and draw the appropriate symbol that many times.

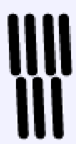
Addition and subtraction with these hieroglyphs has its own quirks. For addition, combine all of the like symbols (all the lines, all the loops, etc.); if there are 10 or more of the same symbol, convert that group of 10 to the next symbol. For subtraction, you can cross out similar-looking symbols and the difference will be what's left over. If there's not enough of one symbol, you may need to split a larger symbol into 10 smaller ones.

Example 3 - Adding and Subtracting Hieroglyphs



Exercise 3

Evaluate each expression. Give your final answer as hieroglyphs.

(a)  + 

(b)  –  (that's a minus)

Egyptian Multiplication

The Ancient Egyptians did not have calculators or even multiplication tables to assist them in calculation. The following method of multiplication relies on knowing your 2's times tables, and mental addition/subtraction.

With this technique, the ancient Egyptians could multiply any two numbers, A and B, where A is greater than B. We'll learn with an example.

To find $A \times B$,

1. Create a chart with two columns, with A and B in the top row on the left and right column respectively.
2. In the first column, start with 1, then keep doubling the number until doubling the last number would give a number larger than A.
3. In the second column, start with B, then double the number directly above until the last row.

Example 4: 26×15**.**

Steps (1) to (3) yield the following chart:

26	15
1	15
2	30
4	60
8	120
16	240

4. Starting at the bottom row of the first column, determine which multiples of 2 sum up to A.
5. Cross out the rows containing multiples of 2 that don't add up to A.
6. Add the remaining numbers in the second column to obtain the product.

Start with the biggest power of 2 that is less than or equal to 26, so 16. Subtracting, we get $26 - 16 = 10$. The next largest power of 2 that is less than or equal to 10 is 8. Subtracting, we get $10 - 8 = 2$. The next power of 2 that is less than or equal to 2 is 2 itself. We cross out the rows with 1 and 4 in the first column.

26	15
~~1~~	~~15~~
2	30
~~4~~	~~60~~
8	120
16	240

Adding the remaining numbers in the second column will give us the answer of $30+120+240 = 390$. Therefore $26 \times 15 = 390$.

Exercise 4

Evaluate the following using the Egyptian Multiplication technique.

(a) 21×18

(b) 𓎆𓎆𓎆𓏺𓏺𓏺 × 𓎆𓎆𓏺𓏺𓏺𓏺

Egyptian Division

The method we will use is similar to Egyptian multiplication. As before we'll look at an example.

Let A be the **dividend** and B be the **divisor**. To find $A \div B$,

1. Create a chart with two columns, with A and B in the top row, in the left and right columns respectively.
2. In the second column, start with B, then keep doubling the number until doubling the last number would give a number larger than A.
3. In the first column, start with 1, then double the number directly above until the last row.

Example 5: $26 \div 4$

Steps (1) to (3) yield the following chart:

26	4
1	4
2	8
4	16

4. Subtract the largest possible number in the right column from A. Repeat this subtraction until all the numbers in the right column are greater than what's leftover. The **remainder** is the "leftover".
5. Cross out the remaining rows (where we did not subtract a number from the second column).
6. Add the remaining numbers in the first column for the **quotient**.

Example 5 continued

Start with $26 - 16 = 10$. We can then do $10 - 8 = 2$. The only remaining number in the right column is 4, which is greater than 2. Hence, 2 is our remainder. Cross out the first row to get

26	4
~~1~~	~~4~~
2	8
4	16

Adding the remaining numbers in the first column will give us a quotient of $2+4 = 6$. Therefore, $26 \div 4 = 6$ remainder 2.

Exercise 5

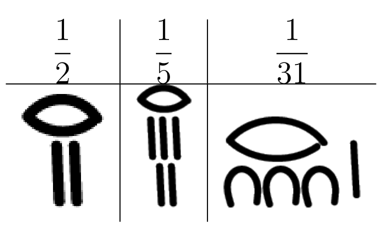
Evaluate $127 \div 13$ using the Egyptian Division technique.

Egyptian Fractions

The Ancient Egyptians used fractions when dividing food or resources, and would only use what are known as **unit fractions**.

A **unit fraction** is a rational number where the **numerator is always 1**. Some examples of unit fractions are then $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{31}$.

To denote fractions, ancient Egyptians would use the "part" hieroglyph,  (mouth), above the other hieroglyphs. Below are the aforementioned unit fractions using this hieroglyph. Notice how the mouth goes over the symbols of greater value.



But how did they represent rational values that *aren't* unit fractions? The Ancient Egyptians would actually represent them as sums of **different** unit fractions.

Example 6

The Egyptians would write $\frac{3}{5}$ as

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$$

However,

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$$

would not count since it is the sum of the same unit fraction (we'll see how they expressed $\frac{2}{3}$ later on).

You might be wondering why would we bother using unit fractions at all. With our mathematical knowledge, rational numbers like $\frac{2}{5}$ and $\frac{97}{100}$ are reasonable as is. Hopefully this next example demonstrates the practicality of unit fractions in another era.

Exercise 6

Suppose you must divide 5 loaves of bread among 8 workers. You must distribute pieces of the bread equally, so we are left with the expression $\frac{5}{8}$. As a decimal this is 0.625, but how much is 62.5% of a loaf of bread?

How would you divide 5 loaves of bread among 8 people equally? Keep in mind that the workers must be convinced that they are not getting less than their coworkers.

What remains is how can we reliably write fractions as a sum of distinct unit fractions. We'll explore more about unit fractions in the Problem Set.